Control Systems I

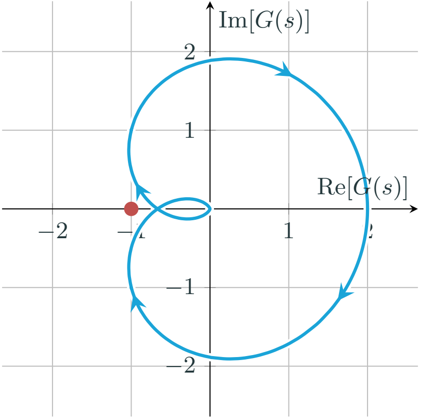
Robustness

Colin Jones

Laboratoire d'Automatique

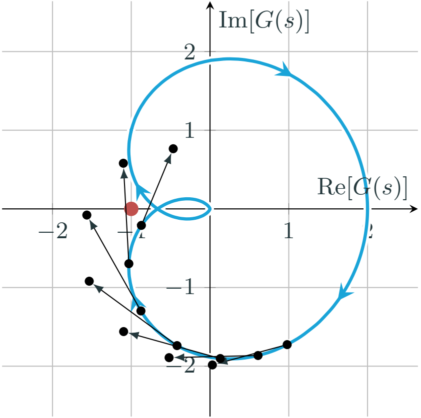
Uncertain Delay

Designed for system $G(s)$, but system in real-world $G'(s) = G(s)e^{-s}$



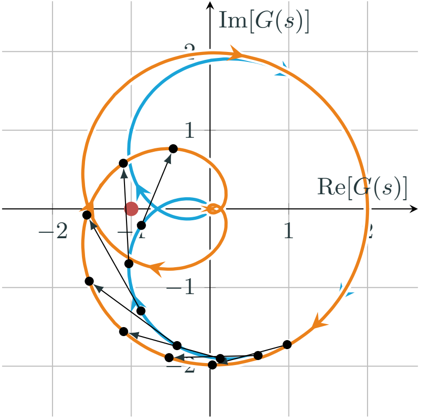
- Impact of delay : Phase delay of $-\omega$
- Original system is stable, delayed system is not

Designed for system $G(s)$, but system in real-world $G'(s) = G(s)e^{-s}$



- Impact of delay : Phase delay of $-\omega$
- Original system is stable, delayed system is not

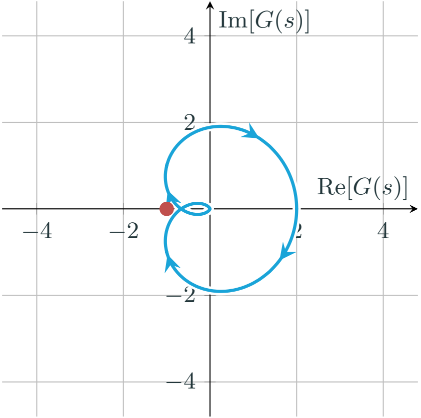
Designed for system $G(s)$, but system in real-world $G'(s) = G(s)e^{-s}$



- Impact of delay : Phase delay of $-\omega$
- Original system is stable, delayed system is not

Example - Uncertain Gain

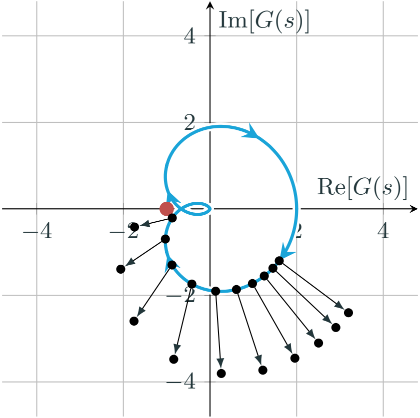
Design for system $G(s)$, but in real-world we get $G'(s) = \alpha G(s)$



- Impact of uncertain gain : Scaling
- Original system is stable, system with larger gain is not

Example - Uncertain Gain

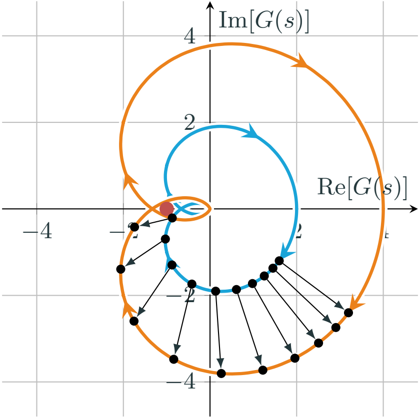
Design for system $G(s)$, but in real-world we get $G'(s) = \alpha G(s)$



- Impact of uncertain gain : Scaling
- Original system is stable, system with larger gain is not

Example - Uncertain Gain

Design for system $G(s)$, but in real-world we get $G'(s) = \alpha G(s)$



- Impact of uncertain gain : Scaling
- Original system is stable, system with larger gain is not

Robustness

The idea: The farther the 'nominal' Nyquist curve is from the -1 point, the more likely the real system will be stable.

The various margins measure distance in terms of some common terms:

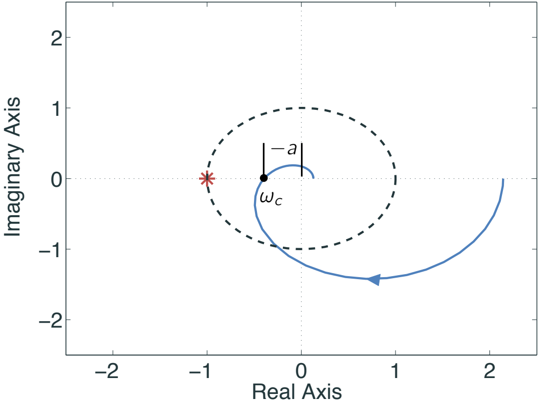
- Uncertain gain
- Uncertain phase
- Uncertain delay

Gain Margin

The ***gain margin*** is the number $1/a$, where

$$a = \min_{\omega} K(j\omega)G(j\omega)$$
$$\text{s.t. } \operatorname{Im} K(j\omega)G(j\omega) = 0$$

i.e., smallest negative crossing of the real axis



- Expressed in decibels

$$-20\log_{10} a > 0$$

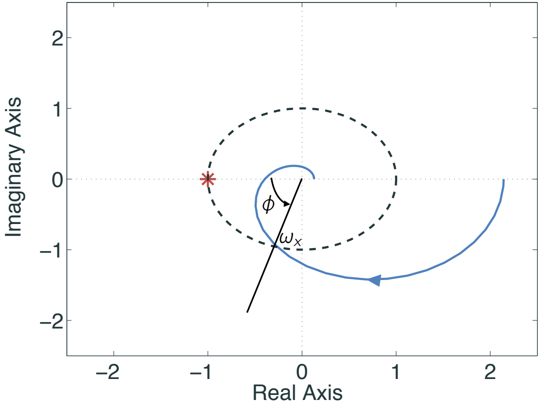
- Amount that gain can increase while stable
- Between 4dB and 12dB generally considered safe
- GM < 1 (0dB) means that the closed-loop system is unstable, GM > 1 (0dB) that it is stable

Phase Margin

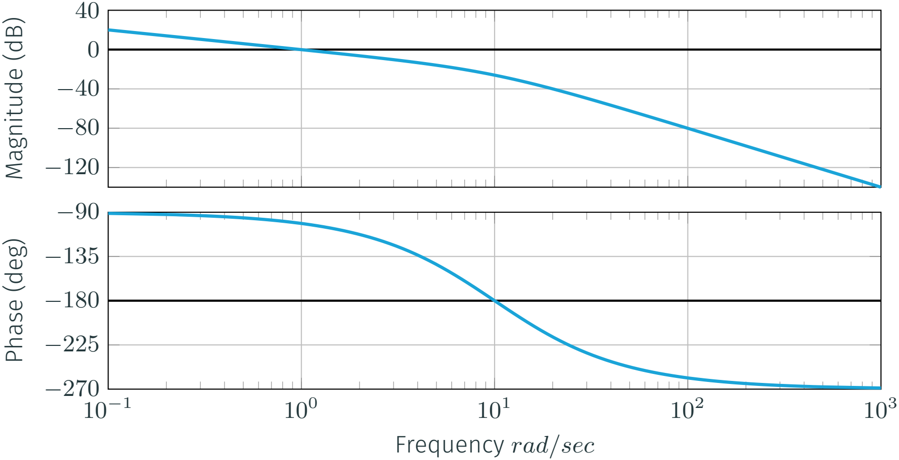
The ***phase margin*** ϕ is the smallest increase in phase that will cause instability

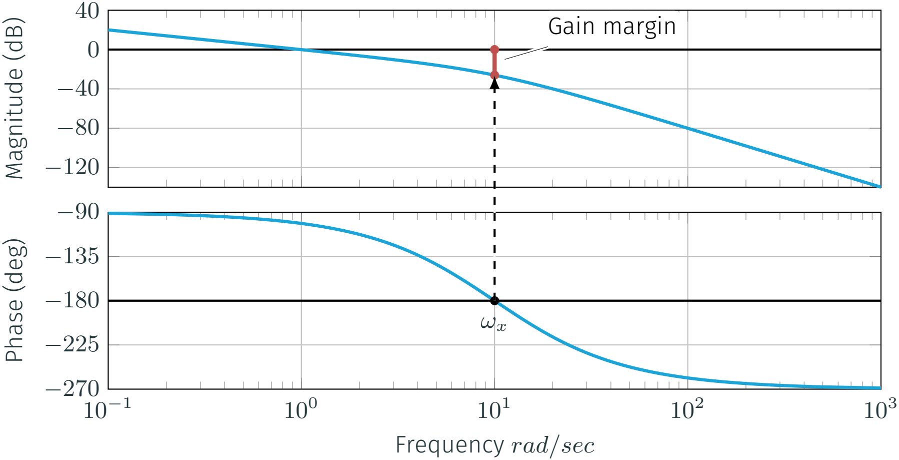
$$\phi = \min_{\phi,\omega} \phi$$
$$\text{s.t. } K(j\omega)G(j\omega)e^{j\phi} = -1$$

- Usually expressed in degrees
- Between 30° and 60° generally considered safe

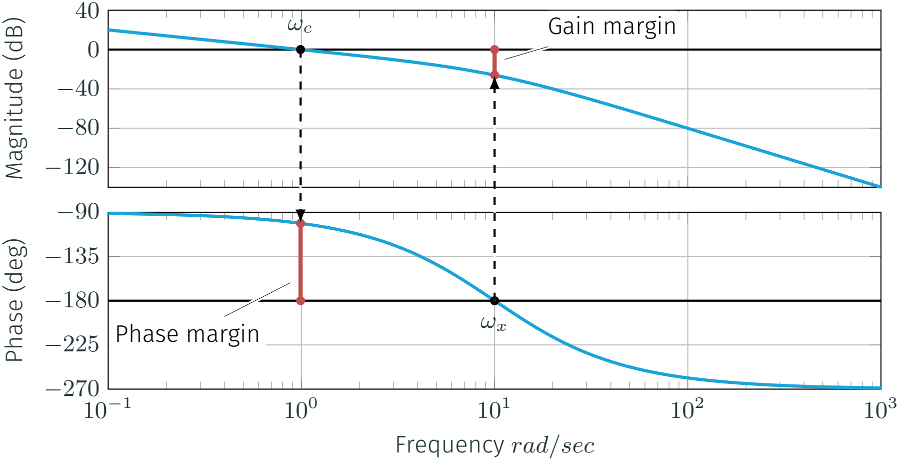


Phase and Gain Margin on Bode Plot




Magnitude (dB)
40
0
−40
−80
−120
Gain margin
Phase (deg)
−90
−135
−180
−225
−270
ω_x
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency rad/sec

Phase and Gain Margin on Bode Plot



Phase and Gain Margin on Bode Plot

Procedure:

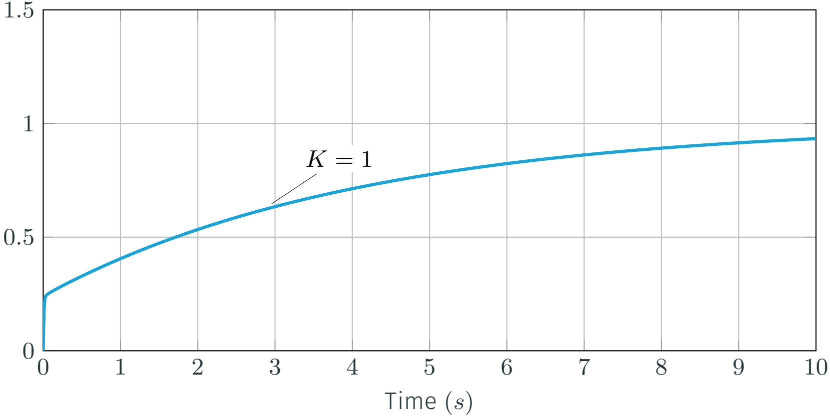
1. Find frequency ω_c that system passes 0dB, from magnitude plot
2. Find frequency ω_x that system passes $-180°$, from the phase plot
3. Gain margin = $-20 \log_{10} |K(j\omega_x)G(j\omega_x)|$dB
4. Phase margin = $\angle K(j\omega_c)G(j\omega_c) - 180$ in degrees

Gain and phase margin positive $\rightarrow$ stable

Example: AFM

$$G(s) = \frac{8.88 \cdot 10^8 (s^2 + 780s + 1.69 \cdot 10^6)}{(s + 3000)(s + 1000)(s + 100)(s^2 + 50s + 6.25 \cdot 10^6)}$$

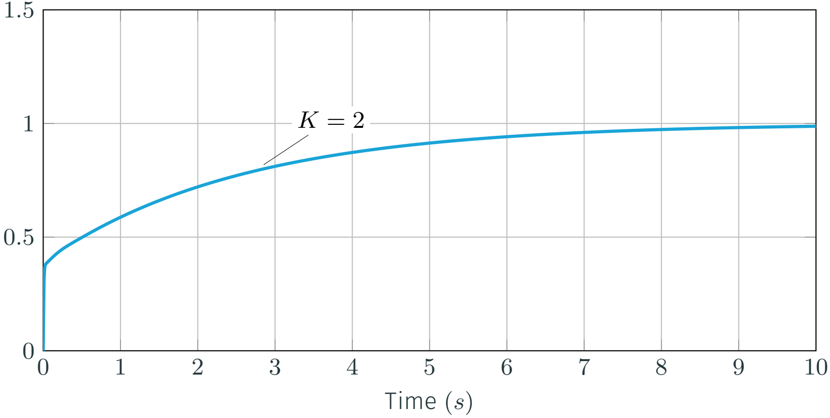
$$K(s) = K \cdot \left(1 + \frac{1}{s}\right)$$



Example: AFM

$$G(s) = \frac{8.88 \cdot 10^8 (s^2 + 780s + 1.69 \cdot 10^6)}{(s+3000)(s+1000)(s+100)(s^2+50s+6.25 \cdot 10^6)}$$

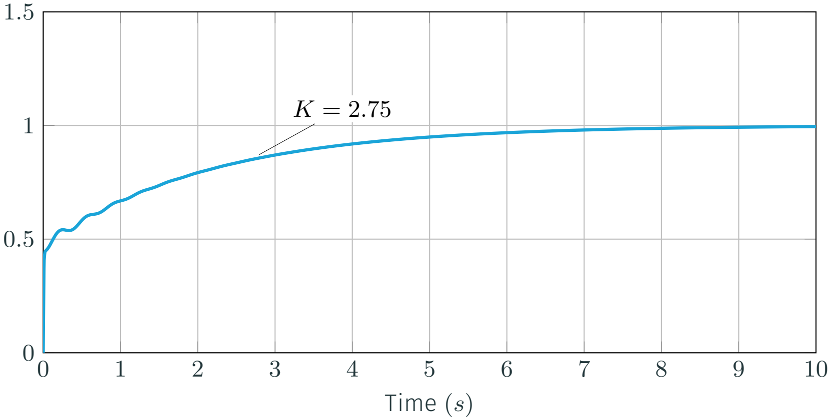
$$K(s) = K \cdot \left(1 + \frac{1}{s}\right)$$



Example: AFM

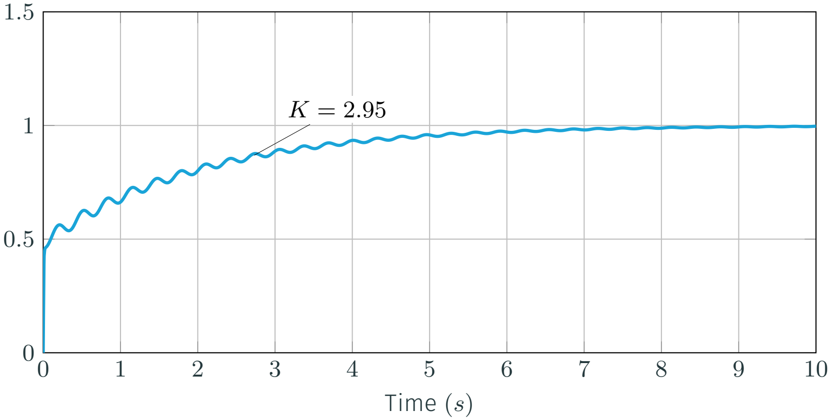
$$G(s) = \frac{8.88 \cdot 10^8 (s^2 + 780s + 1.69 \cdot 10^6)}{(s+3000)(s+1000)(s+100)(s^2 + 50s + 6.25 \cdot 10^6)}$$

$$K(s) = K \cdot \left(1 + \frac{1}{s}\right)$$



$$G(s) = \frac{8.88 \cdot 10^8 (s^2 + 780s + 1.69 \cdot 10^6)}{(s + 3000)(s + 1000)(s + 100)(s^2 + 50s + 6.25 \cdot 10^6)}$$

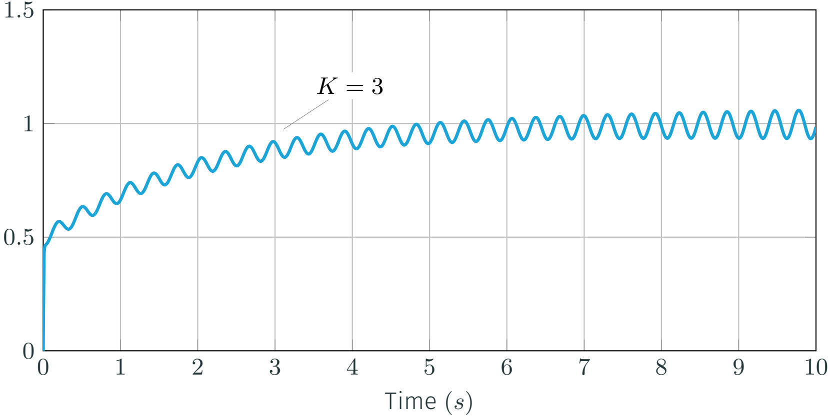
$$K(s) = K \cdot \left(1 + \frac{1}{s}\right)$$



Example: AFM

$$G(s) = \frac{8.88 \cdot 10^8 (s^2 + 780s + 1.69 \cdot 10^6)}{(s+3000)(s+1000)(s+100)(s^2+50s+6.25 \cdot 10^6)}$$

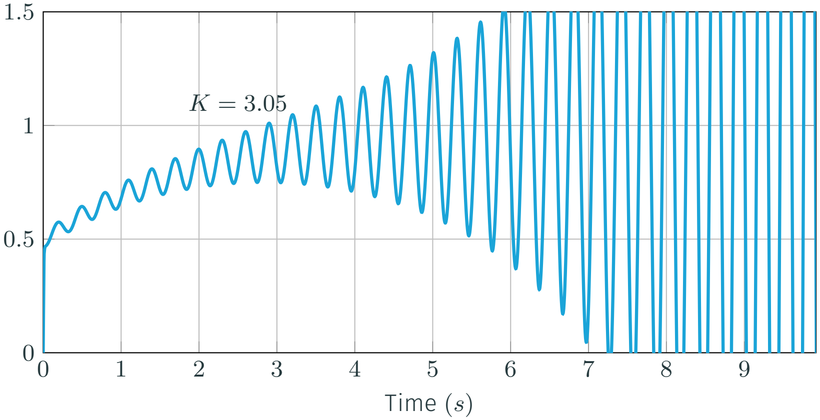
$$K(s) = K \cdot \left(1 + \frac{1}{s}\right)$$



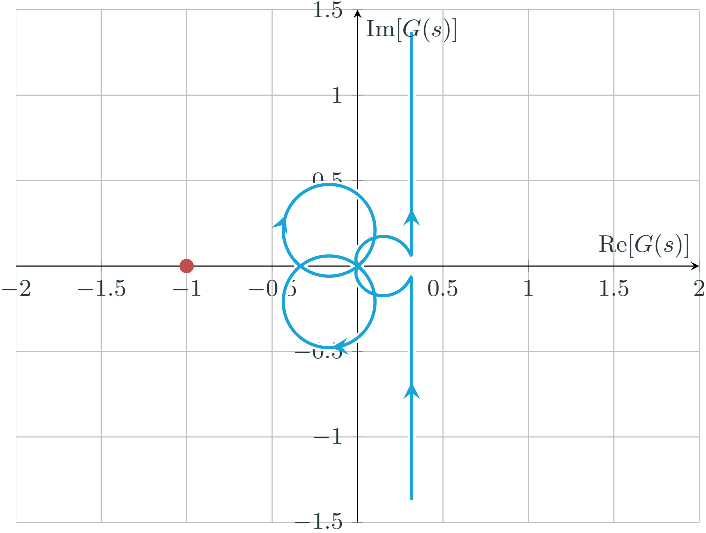
Example: AFM

$$G(s) = \frac{8.88 \cdot 10^8 (s^2 + 780s + 1.69 \cdot 10^6)}{(s+3000)(s+1000)(s+100)(s^2+50s+6.25 \cdot 10^6)}$$

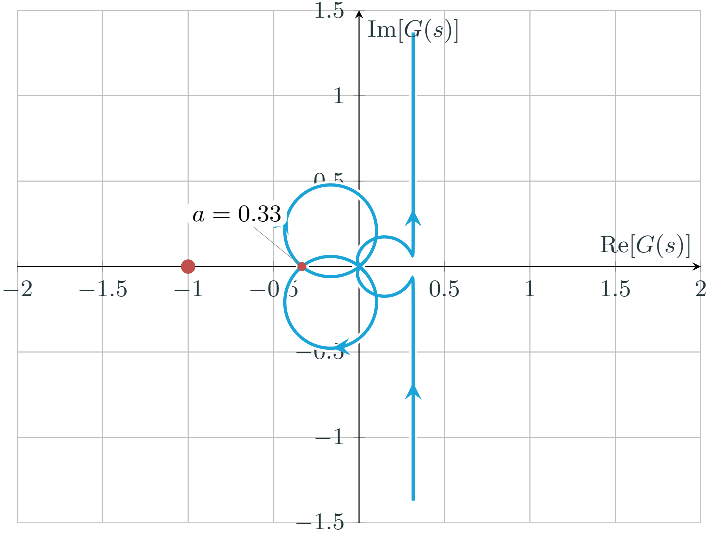
$$K(s) = K \cdot \left(1 + \frac{1}{s}\right)$$



System becomes unstable somewhere between $K = 3$ and $K = 3.05$

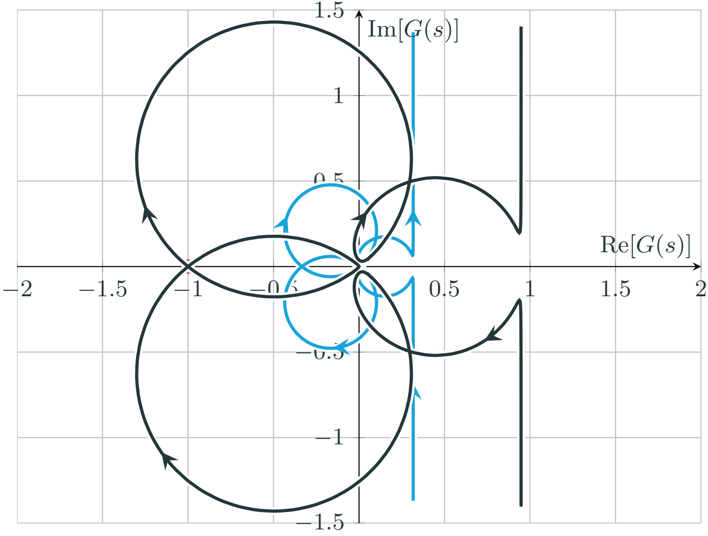


1.5
1
0.5
−0.5
−1
−1.5
−2
−1.5
−1
−0.5
0.5
1
1.5
2
Im[G(s)]
Re[G(s)]

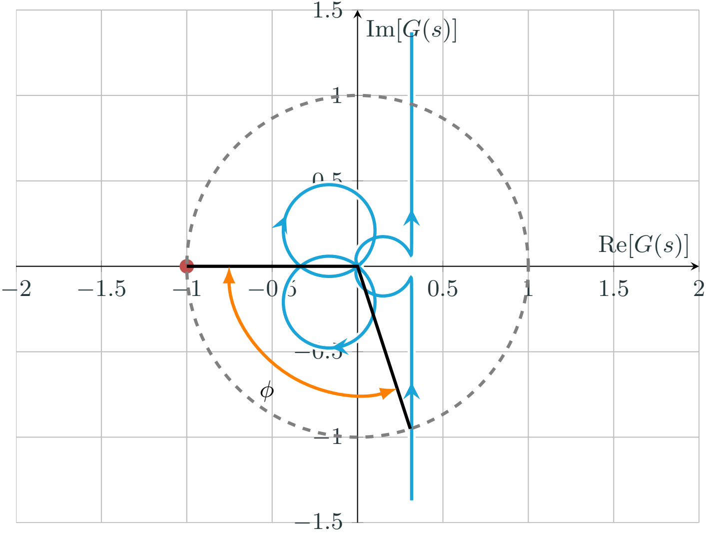


Gain margin = $-20\log_{10} a = 9.63dB$

System becomes unstable at a gain of $1/a = 3.03$

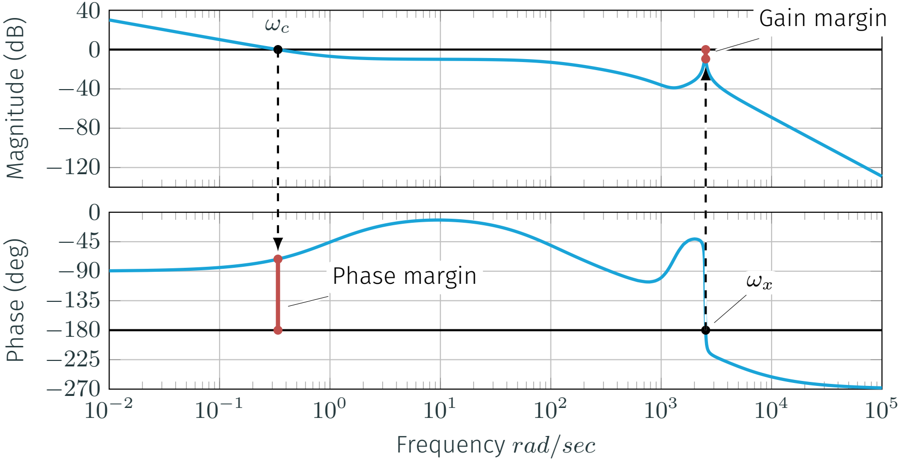


1.5
1
0.5
−0.5
−1
−1.5
−2
−1.5
−1
−0.5
0.5
1
1.5
2
Im[G(s)]
Re[G(s)]

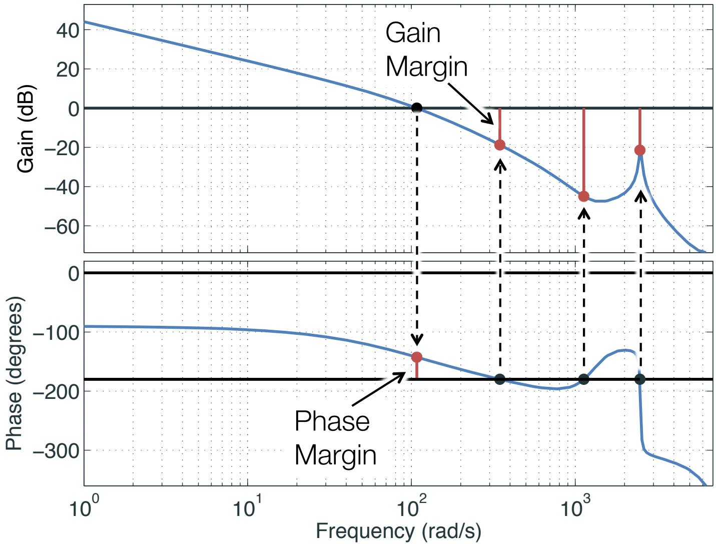


Phase margin = 108 degrees

Gain and Phase Margins : Bode



Gain and Phase Margins : Bode



Choose the 'first to be unstable' if there are multiple crossings

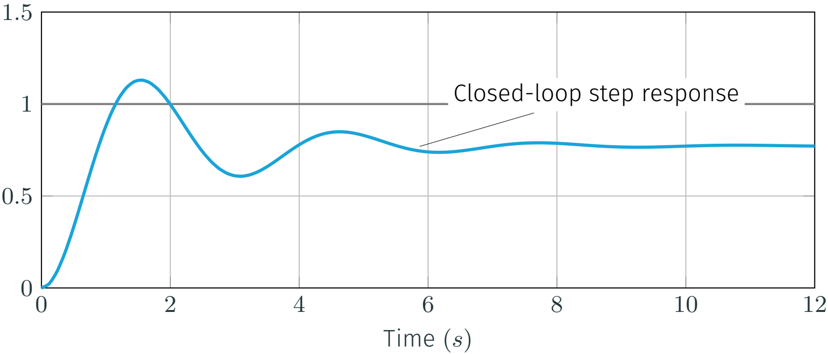
Margins - Summary

- Margins measure how far from stability the closed-loop system is in terms of a single uncertain parameter
  - Phase
  - Gain
  - Delay
- Many applications specify minimum phase and gain margins for safety
- In later lectures we will look at dynamic controllers that ***shape*** the frequency response so that we have good margins

Steady State Errors

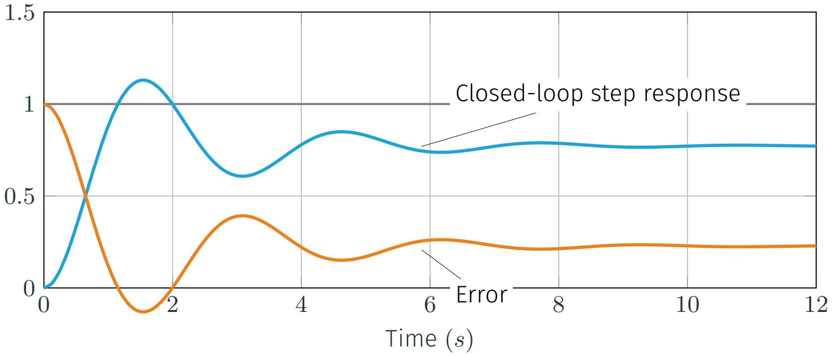
Steady-State Offset

$$G(s) = \frac{3.4}{s^2 + s + 1} \qquad K(s) = 1$$



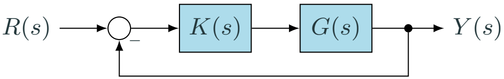
Steady-State Offset

$$G(s) = \frac{3.4}{s^2 + s + 1} \qquad K(s) = 1$$



Stable does not mean that the output tracks the reference!

Steady-State Offset



$$E(s) = \frac{1}{1+K(s)G(s)}R(s)$$
$$= \mathcal{S}(s)R(s)$$

Theorem

Final Value Theorem

$$\lim_{t\to\infty} w(t) = \lim_{s\to 0} sW(s)$$

If poles of $sW(s)$ are in the left half plane

If $r(t) = 1, t \geq 0$, then $R(S) = 1/s$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\frac{1}{1+K(s)G(s)} \cdot \frac{1}{s} = \lim_{s\to 0} \frac{1}{1+K(s)G(s)} =???$$

- Not infinite because $K(s)G(s) \neq -1$
- Called this the ***steady-state offset*** or ***steady-state error***
- Conditions for the steady-state offset to be zero?

System Type and Open-Loop Steady-State Gain

Suppose that the open-loop transfer function has q poles at $s = 0$

$$K(s)G(s) = \frac{B(s)}{s^q A(s)}$$

where A and B are polynomials.

The number q is called the ***type*** or the ***class*** of the open-loop system.

The ***open-loop steady-state gain*** of the system $K(s)G(s)$ is

$$\gamma := \lim_{s \to 0} s^q K(s)G(s) = \frac{B(0)}{A(0)}$$

Type 0 System Response to a Step Command

Suppose we have a type 0 system

$$K(s)G(s) = \frac{B(s)}{A(s)}$$

and we apply a step input

$$R(s) = \frac{1}{s}$$

Type 0 System Response to a Step Command

Suppose we have a type 0 system

$$K(s)G(s) = \frac{B(s)}{A(s)}$$

and we apply a step input

$$R(s) = \frac{1}{s}$$

The steady-state error for the closed-loop system will be

$$\begin{aligned} \lim_{t \to \infty} e(t) &= \lim_{s \to 0} s \frac{1}{1 + K(s)G(s)} \cdot \frac{1}{s} \\ &= \frac{1}{1 + \gamma} \end{aligned}$$

Type 1 System Response to a Step Command

Suppose we have a type 1 system

$$K(s)G(s) = \frac{B(s)}{sA(s)}$$

and we apply a step input

$$R(s) = \frac{1}{s}$$

Type 1 System Response to a Step Command

Suppose we have a type 1 system

$$K(s)G(s) = \frac{B(s)}{sA(s)}$$

and we apply a step input

$$R(s) = \frac{1}{s}$$

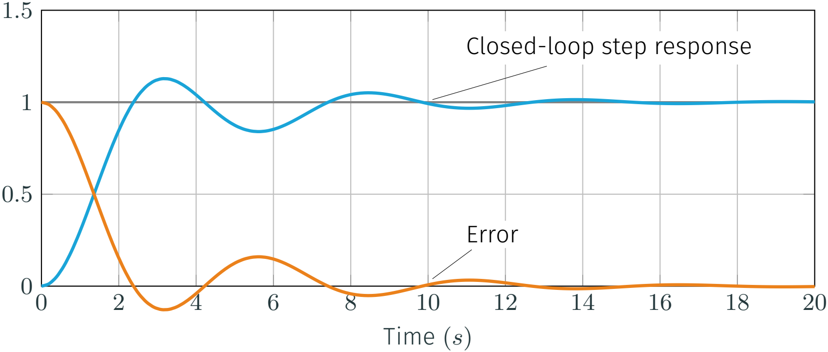
The steady-state error for the closed-loop system will be

$$\begin{aligned}\lim_{t\to\infty} e(t) &= \lim_{s\to 0} s\frac{1}{1+K(s)G(s)} \cdot \frac{1}{s} \\ &= \lim_{s\to 0} \frac{s}{s+\frac{B(s)}{A(s)}} \\ &= 0\end{aligned}$$

The steady-state error will be zero for all step inputs and systems!

Type 1 System Response to a Step Command

$$G(s) = \frac{3.4}{s^2 + s + 1} \qquad K(s) = \frac{1}{5}\left(1 + \frac{1}{s}\right)$$



Zero steady-state offset

Type 1 System Response to a Ramp Command

Suppose we have a type 1 system

$$K(s)G(s) = \frac{B(s)}{sA(s)}$$

and we apply a ***ramp input*** $r(t) = t$

$$R(s) = \frac{1}{s^2}$$

Type 1 System Response to a Ramp Command

Suppose we have a type 1 system

$$K(s)G(s) = \frac{B(s)}{sA(s)}$$

and we apply a ***ramp input*** $r(t) = t$

$$R(s) = \frac{1}{s^2}$$

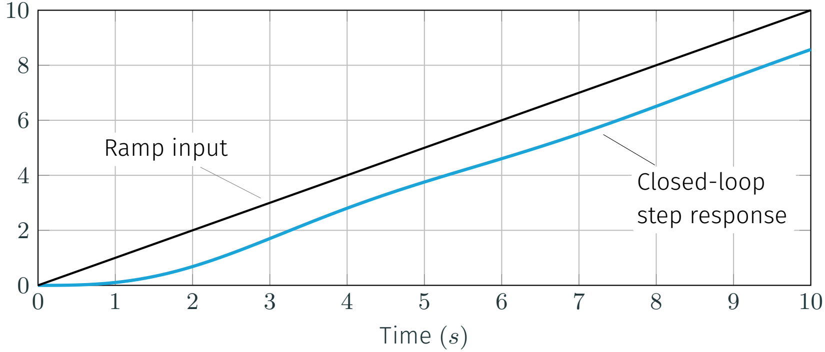
The steady-state error for the closed-loop system will be

$$\begin{aligned}
\lim_{t \to \infty} e(t) &= \lim_{s \to 0} s \cdot \frac{1}{1 + K(s)G(s)} \cdot \frac{1}{s^2} \\
&= \lim_{s \to 0} \frac{1}{1 + \frac{B(s)}{sA(s)}} \cdot \frac{1}{s} \\
&= \frac{1}{\gamma}
\end{aligned}$$

Non-zero steady-state error.

Type 1 System Response to a Ramp Command

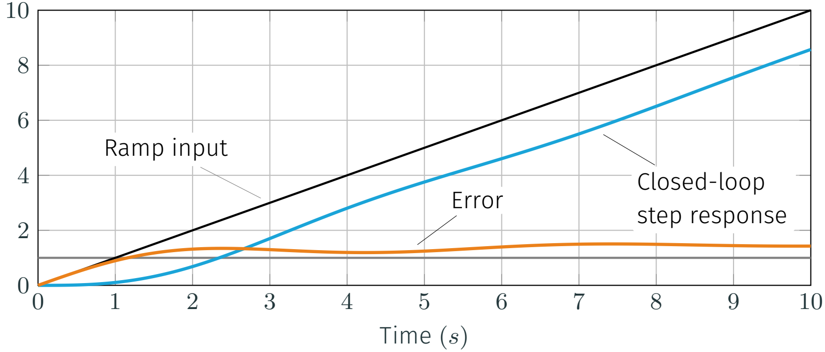
$$G(s) = \frac{3.4}{s^2 + s + 1} \qquad K(s) = \frac{1}{5}\left(1 + \frac{1}{s}\right)$$



Constant steady-state offset

Type 1 System Response to a Ramp Command

$$G(s) = \frac{3.4}{s^2 + s + 1} \qquad K(s) = \frac{1}{5}\left(1 + \frac{1}{s}\right)$$



Constant steady-state offset

Type 1 System Response to a Parabolic Command

Suppose we have a type 1 system

$$K(s)G(s) = \frac{B(s)}{sA(s)}$$

and we apply a ***parabolic input*** $r(t) = t^2$

$$R(s) = \frac{2}{s^3}$$

Type 1 System Response to a Parabolic Command

Suppose we have a type 1 system

$$K(s)G(s) = \frac{B(s)}{sA(s)}$$

and we apply a ***parabolic input*** $r(t) = t^2$

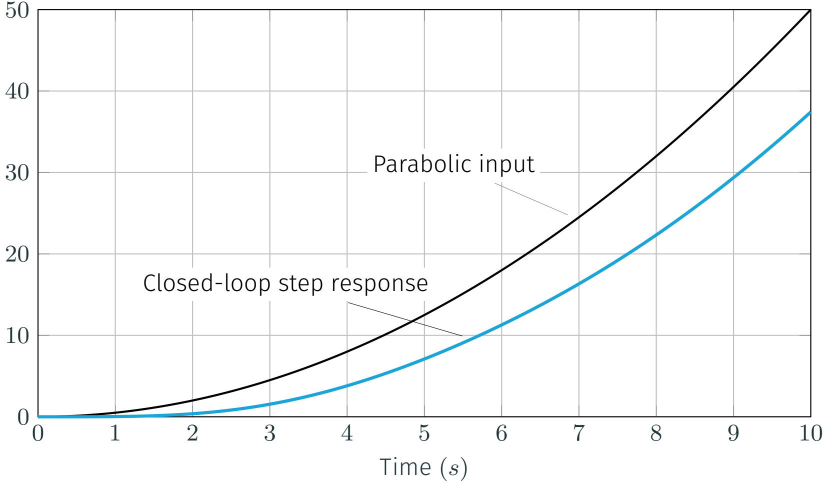
$$R(s) = \frac{2}{s^3}$$

The error is

$$\begin{aligned} E(s) &= \frac{1}{1 + K(s)G(s)} \cdot \frac{2}{s^3} \\ &= \frac{sB(s)}{sA(s) + B(s)} \cdot \frac{2}{s^3} \\ &= \frac{B(s)}{sA(s) + B(s)} \cdot \frac{2}{s^2} \end{aligned}$$

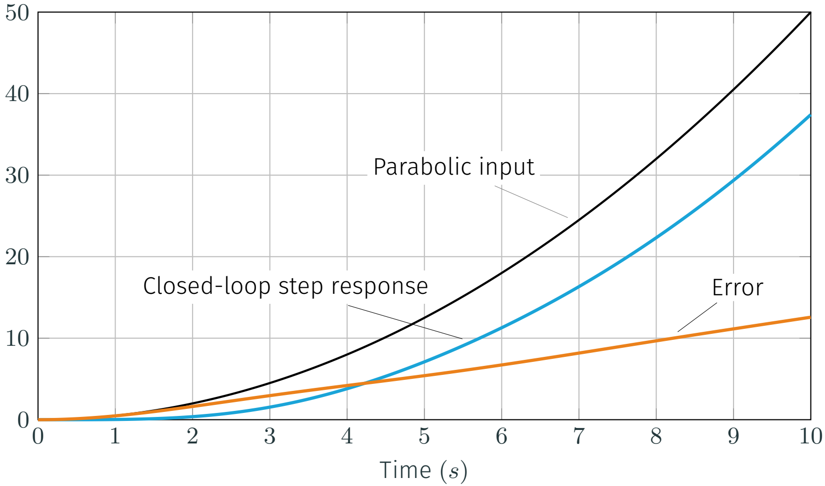
Cannot apply final-value theorem because there is more than one pole at 0
$\rightarrow$ Implies that the error is either unbounded, or oscillates

Type 1 System Response to a Parabolic Command



Infinite steady-state offset

Type 1 System Response to a Parabolic Command



Infinite steady-state offset

Steady-State Errors by System Type

Type	$r(t) = 1$	$r(t) = t$	$r(t) = t^2$
0	$\dfrac{1}{1+\gamma}$	∞	∞
1	0	$\dfrac{1}{\gamma}$	∞
2	0	0	$\dfrac{1}{\gamma}$

Basic idea: Must have more integrators than the signal you're trying to track

Why not just add hundreds of integrators, and track anything?!

Steady-State Errors by System Type - by the Book

The book uses a slightly different notation:

Type	$r(t) = 1$ Step (position)	$r(t) = t$ Ramp (velocity)	$r(t) = t^2$ Parabola (acceleration)
0	$\frac{1}{1+K_p}$	∞	∞
1	0	$\frac{1}{K_v}$	∞
2	0	0	$\frac{1}{K_a}$

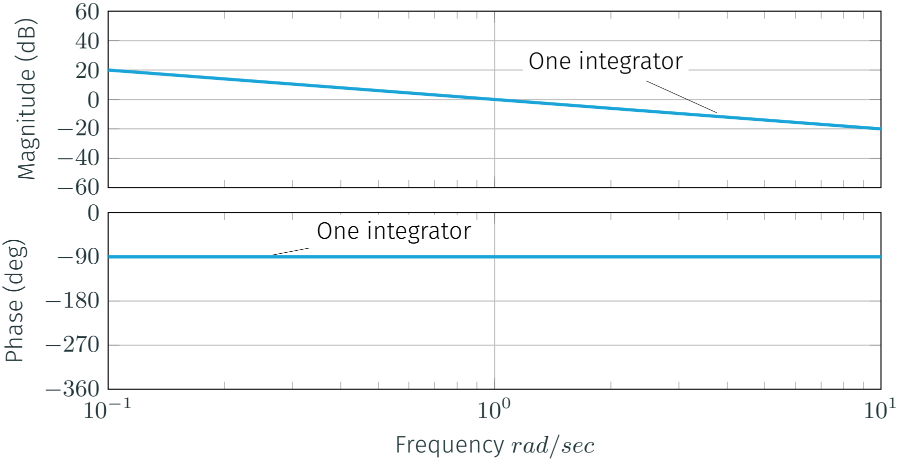
$$K_p = \lim_{s \to 0} K(s)G(s) \qquad \text{Type 0}$$

$$K_v = \lim_{s \to 0} sK(s)G(s) \qquad \text{Type 1}$$

$$K_a = \lim_{s \to 0} s^2 K(s)G(s) \qquad \text{Type 2}$$

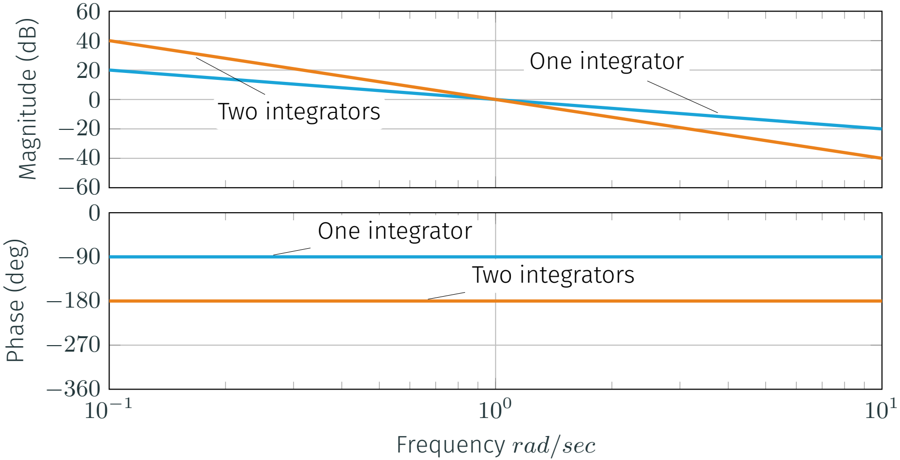
The book differentiates constants between position, velocity and acceleration. We just use γ for all of them.

$$\frac{1}{s^q} \rightarrow (j\omega)^{-q}$$

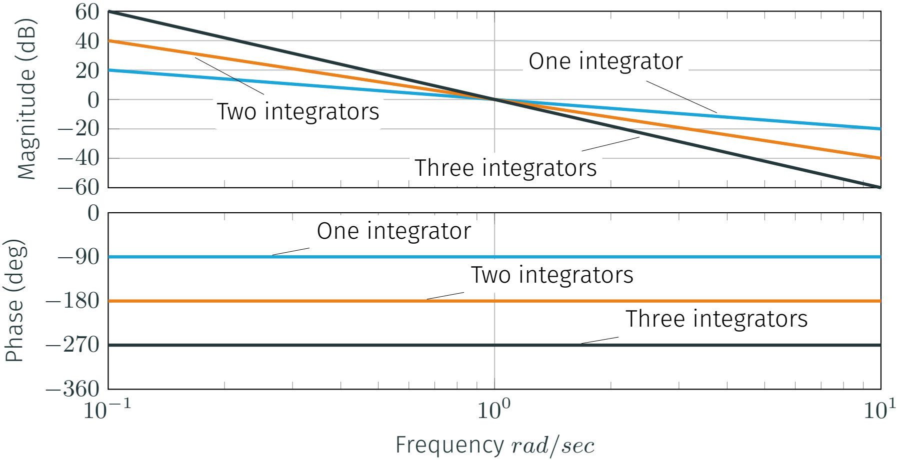


Negative Impact of Adding Integrators

$$\frac{1}{s^q} \rightarrow (j\omega)^{-q}$$



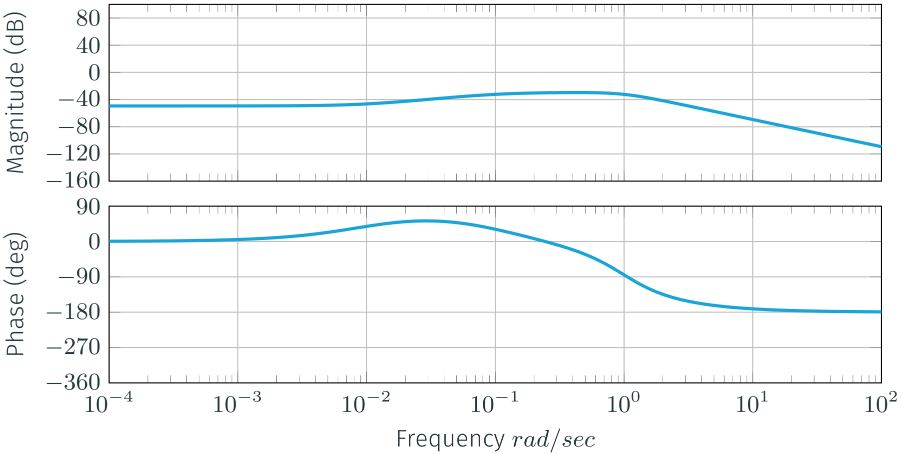
$$\frac{1}{s^q} \rightarrow (j\omega)^{-q}$$



Negative Impact of Adding Integrators

How do the two systems differ in terms of stability margins?

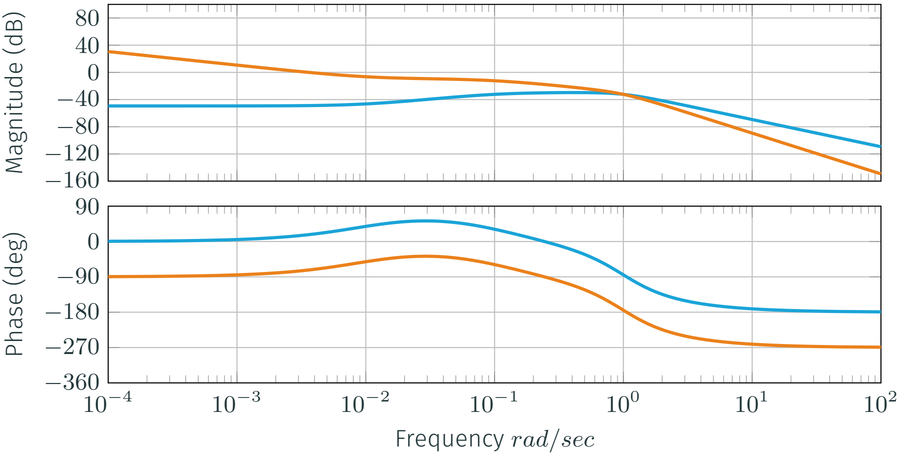
$$G(s) \quad \text{vs} \quad \frac{G(s)}{s^q}$$



Negative Impact of Adding Integrators

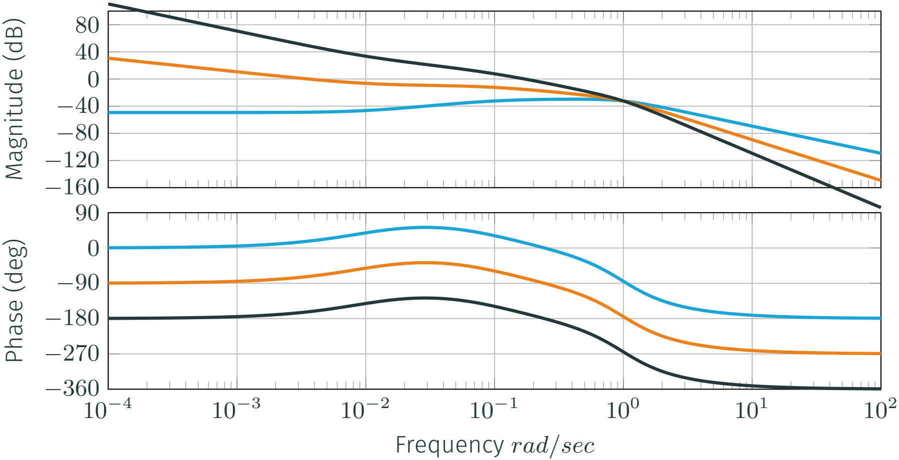
How do the two systems differ in terms of stability margins?

$$G(s) \qquad \text{vs} \qquad \frac{G(s)}{s^q}$$



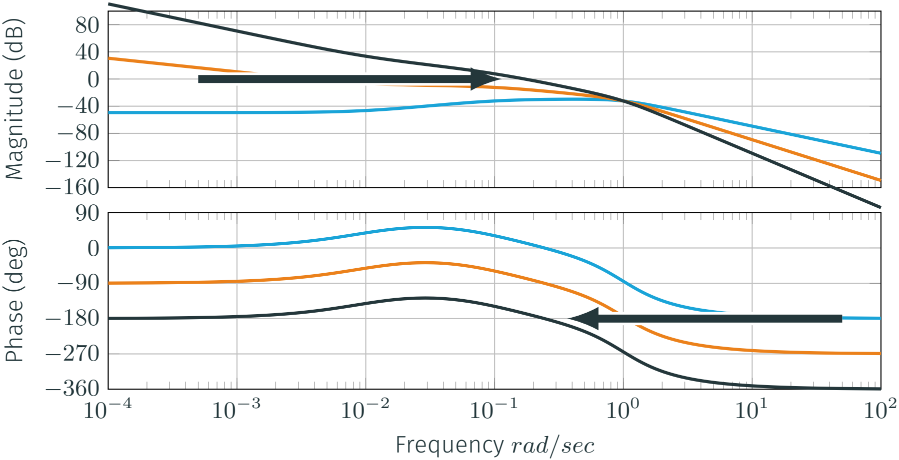
How do the two systems differ in terms of stability margins?

$$G(s) \quad \text{vs} \quad \frac{G(s)}{s^q}$$

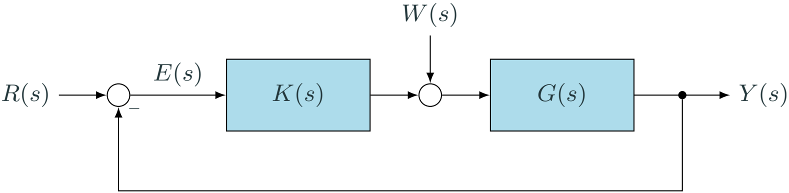


How do the two systems differ in terms of stability margins?

$$G(s) \quad \text{vs} \quad \frac{G(s)}{s^q}$$



Constant Disturbance Rejection



System response with respect to a disturbance $w(t)$ is:

$$E(s) = \frac{G(s)}{1 + K(s)G(s)} W(s)$$

What are the conditions for zero steady-state offset with respect to a constant disturbance?

Constant Disturbance Rejection

Suppose the system has q integrators, and K has none

$$G(s) = \frac{B(s)}{s^q A(s)}$$

Constant Disturbance Rejection

Suppose the system has q integrators, and K has none

$$G(s) = \frac{B(s)}{s^q A(s)}$$

The error is:

$$\begin{aligned} E(s) &= \frac{G(s)}{1 + K(s)G(s)} W(s) \\ &= \frac{B(s)}{s^q A(s) + K(s)B(s)} \cdot \frac{1}{s} \end{aligned}$$

Then the steady-state error is:

$$\lim_{s \to 0} sE(s) = \frac{1}{K(0)}$$

Integrators in the system do not reject disturbances!

Integrators in the Controller

Suppose the controller K has r integrators

$$K(s) = \frac{S(s)}{s^r R(s)}$$

Integrators in the Controller

Suppose the controller K has r integrators

$$K(s) = \frac{S(s)}{s^r R(s)}$$

The error is:

$$\begin{aligned} E(s) &= \frac{B(s)}{s^q A(s) + K(s)B(s)} \cdot \frac{1}{s} \\ &= s^r \frac{B(s)R(s)}{s^r s^q A(s)R(s) + S(s)B(s)} \cdot \frac{1}{s} \end{aligned}$$

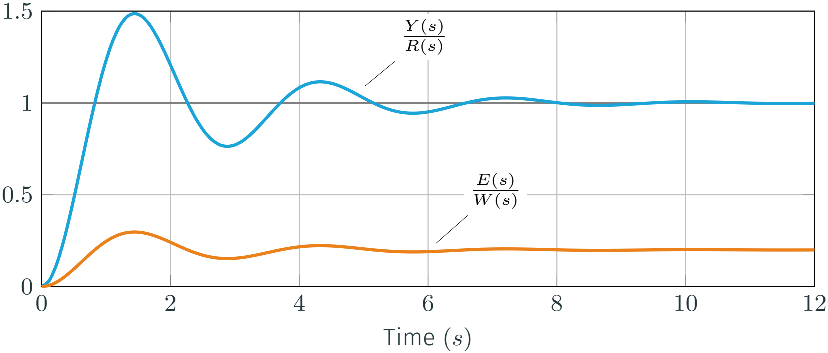
The controller's integrators do reject the disturbance:

- One pole at $0 \to$ rejects constant disturbance
- Two poles at $0 \to$ rejects ramp disturbance
- $\vdots$
- etc

Example

$$G(s) = \frac{1}{s(s+1)} \qquad\qquad K(s) = 5$$

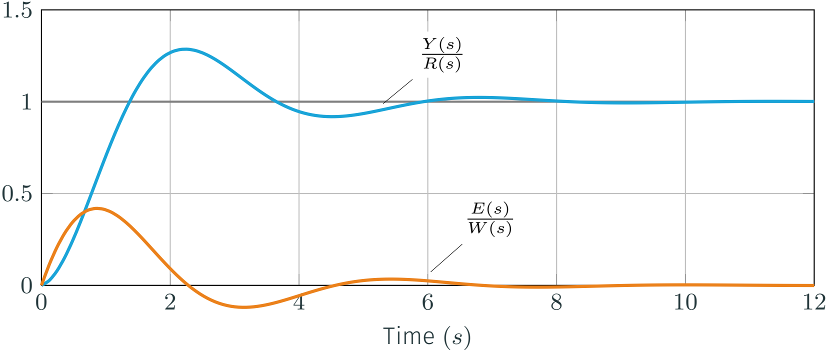
- System has an integrator
- Controller doesn't



Example

$$G(s) = \frac{1}{s+1} \qquad\qquad K(s) = 0.1 + \frac{2.2}{s}$$

- System doesn't have an integrator
- Controller does (PI)



Waterbed Effect

Dynamic Disturbance Rejection

We want to reject 'complex' signals.

Consider a sinusoid

$$r(t) = \sin(\omega t) \ \ ,$$

or a mix of sinusoids.

The rejection of these signals, or the sensitivity to them, is given by the sensitivity function

$$\left| \frac{1}{1 + K(j\omega)G(j\omega)} \right| = |\mathcal{S}(j\omega)|$$

Theorem Bode's Integral Formula

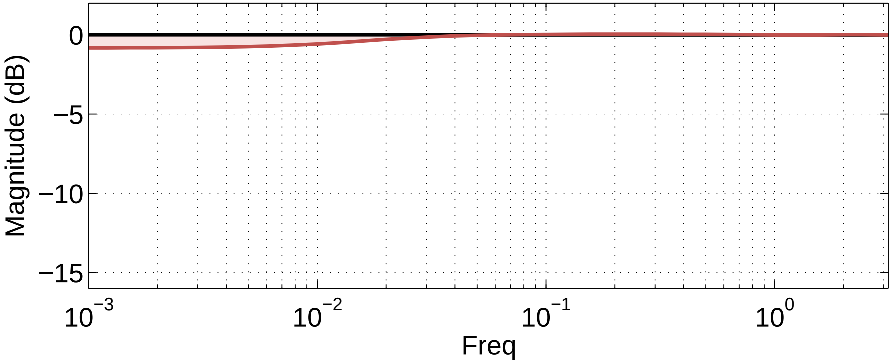
Assume we have a closed-loop stable system with open-loop unstable poles p_i, $i = 1, 2, \ldots, P$ and a strictly proper open-loop transfer function $K(s)G(s)$. The sensitivity function satisfies the condition

$$\int_0^\infty \log |\mathcal{S}(j\omega)| d\omega = \pi \sum_{i=1}^{P} \mathrm{Re}(p_i)$$

This is a fundamental limit on how well the system can perform:

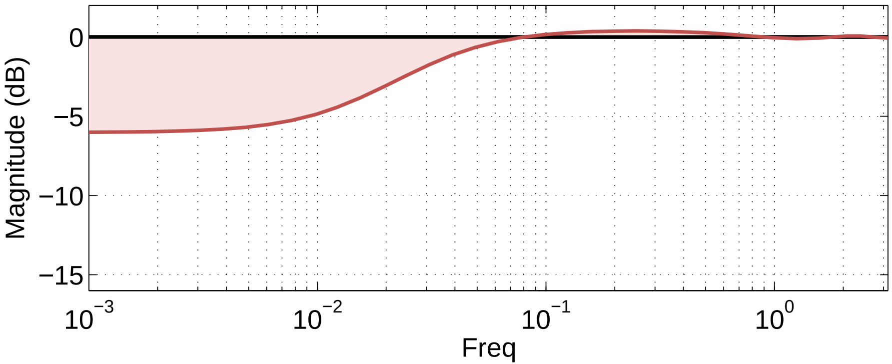
- If we damp noise for some frequencies $|\mathcal{S}(j\omega)| < 1$, then we must ***amplify*** it $|\mathcal{S}(j\omega)| > 1$ at others!
- This is called the ***waterbed effect***
- Harder to get good disturbance rejection behaviour out of unstable systems (those with many unstable poles)

$$K(s)G(s) = K_p \frac{s^2 - 133.3s + 5926}{(s+1)(s^2 + 133.3s + 5926)}$$



$K_p = 0.1$ $\max_{\omega} \mathcal{S}(j\omega) = 0.04dB$

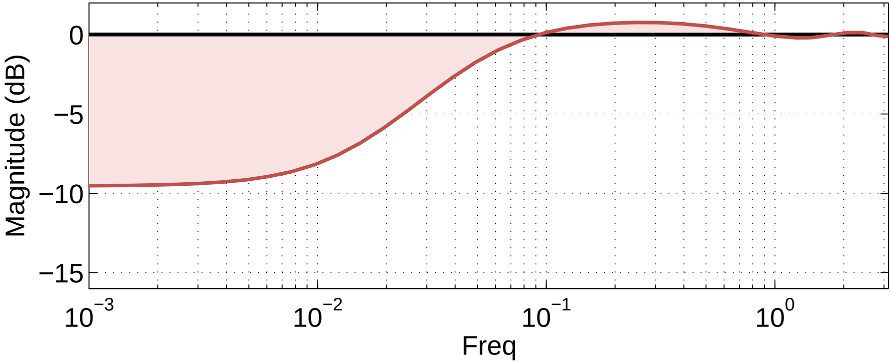
$$K(s)G(s) = K_p \frac{s^2 - 133.3s + 5926}{(s+1)(s^2 + 133.3s + 5926)}$$



$K_p = 1$ $\qquad$ $\max_{\omega} \mathcal{S}(j\omega) = 0.4dB$

Example

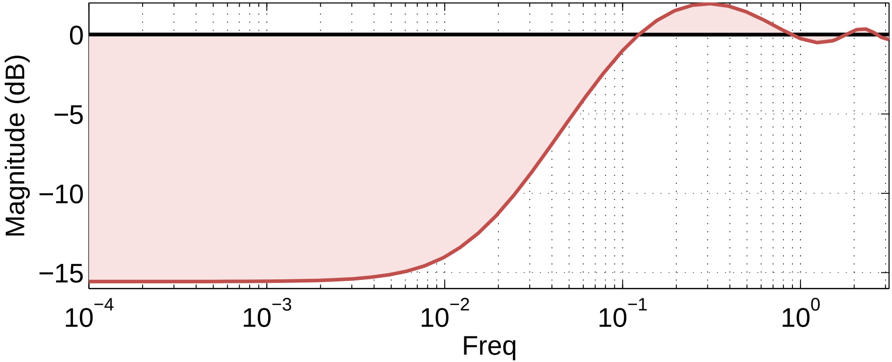
$$K(s)G(s) = K_p \frac{s^2 - 133.3s + 5926}{(s+1)(s^2 + 133.3s + 5926)}$$



$K_p = 2$ $\qquad\qquad$ $\max_{\omega} \mathcal{S}(j\omega) = 0.8dB$

Example

$$K(s)G(s) = K_p \frac{s^2 - 133.3s + 5926}{(s+1)(s^2 + 133.3s + 5926)}$$



$K_p = 5$ $\max_{\omega} \mathcal{S}(j\omega) = 2dB$

Summary

Robustness: The farther the 'nominal' Nyquist curve is from the -1 point, the more likely the real system will be stable.

"Margins" measure how far your system is from unstable

- Gain margin
- Phase margin
- Delay margin

Steady-state offset

- Need to have as many integrators in your controller as are in the signal to track / reject if you want zero steady-state error

Waterbed effect

- There is a fundamental limit to how well a controller can work
- Cannot improve noise rejection / tracking at all frequencies